Economics 103 – Statistics for Economists

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Lecture 15

Last Time

Confidence Interval for Population Mean:

$$\boxed{\bar{X}_n \pm \texttt{qnorm}(1-\alpha/2) \times \sigma/\sqrt{n}}$$

Based on Assumptions:

1. The population standard deviation σ was known.
2. The population is normally distributed (bell-shaped).

Today

- What if population is normal but σ is unknown?
- We will show that $\frac{\bar{X}_n - \mu}{S/\sqrt{n}} \sim t(n-1)$

We Don't know σ. What to use instead?

$$\boxed{\bar{X}_n \pm \texttt{qnorm}(1-\alpha/2) \times \sigma/\sqrt{n}}$$

What about Sample Standard Deviation S?

$$P\left(-2 \leq \frac{\bar{X}_n - \mu}{S/\sqrt{n}} \leq 2\right) = 0.95 \text{ ???}$$

Not Quite!

Although $(\bar{X}_n - \mu)/(\sigma/\sqrt{n}) \sim N(0,1)$, $S \neq \sigma$. In fact, S is an estimator of σ so it is a random variable!

What is the sampling distribution?

Suppose $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$

$$\boxed{\frac{\bar{X}_n - \mu}{S/\sqrt{n}} \sim \text{ ???}}$$

First Step

What is the sampling distribution of S?

What is the Distribution?

Suppose $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. What is the distribution of this sum?

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2$$

(a) $\chi^2(n)$

(b) $N(\mu, \sigma^2)$

(c) $N(0, 1)$

(d) $N(\mu, \sigma^2/n)$

(e) $\chi^2(1)$

Towards the Sampling Dist. of S

If $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, then

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(n)$$

Now:

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 = \left(\frac{n-1}{\sigma^2} \right) \left[\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \mu)^2 \right] \sim \chi^2(n)$$

Anything look familiar?

Sampling Distribution of Sample Variance

Suppose $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. Then whereas

$$\left(\frac{n-1}{\sigma^2}\right)\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\mu)^2\right] \sim \chi^2(n)$$

Replacing μ with $\bar{X}$ "loses" a degree of freedom

$$\left(\frac{n-1}{\sigma^2}\right)\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2\right] = \left(\frac{n-1}{\sigma^2}\right) S^2 \sim \chi^2(n-1)$$

We will use this fact to work out the sampling distribution of $\sqrt{n}(\bar{X}_n - \mu)/S$

What is the Sampling Distribution of $\sqrt{n}(\bar{X}_n - \mu)/S$?

This slide is just algebra:

$$\begin{aligned}
\frac{\bar{X}_n - \mu}{S/\sqrt{n}} &= \frac{\bar{X}_n - \mu}{S/\sqrt{n}} \cdot \left(\frac{\sigma/\sqrt{n}}{\sigma/\sqrt{n}}\right) = \left(\frac{\bar{X}_n - \mu}{\color{red}{\sigma/\sqrt{n}}}\right)\left(\frac{\sigma/\sqrt{n}}{\color{red}{S/\sqrt{n}}}\right) \\
&= \left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)\left(\frac{\sigma}{S}\right) = \left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)\left(\sqrt{\frac{n-1}{n-1}} \cdot \sqrt{\frac{\sigma^2}{S^2}}\right) \\
&= \left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)\left(\sqrt{\frac{(n-1)\sigma^2}{(n-1)S^2}}\right) \\
&= \frac{\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)}{\sqrt{\left[\frac{(n-1)S^2}{\sigma^2}\right]/(n-1)}}
\end{aligned}$$

Distribution of numerator, $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$

Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$ and $\bar{X}_n$ is ths sample mean.

Then the sampling distribution of $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$ is

(a) $\chi^2(n)$

(b) $\chi^2(n-1)$

(c) $N(\mu, \sigma^2)$

(d) $N(0, 1)$

(e) $N(\mu, \sigma^2/n)$

$N(0, 1)$

Distribution of $(n-1)S^2/\sigma^2$

Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$ and S^2 is the sample variance. Then the sampling distribution of $(n-1)S^2/\sigma^2$ is

(a) $\chi^2(n)$

(b) $\chi^2(n-1)$

(c) $N(\mu, \sigma^2)$

(d) $N(0, 1)$

(e) $N(\mu, \sigma^2/n)$

From slide 7, $\chi^2(n-1)$

What is the Sampling Distribution?

Suppose $Z \sim N(0,1)$ independent of $Y \sim \chi^2(n-1)$. Then the sampling distribution of $Z/\sqrt{Y/(n-1)}$ is

(a) $t(n)$

(b) $t(n-1)$

(c) $\chi^2(n)$

(d) $N(0,1)$

(e) $F(n, n-1)$

From end of lecture 11, $t(n-1)$

What is the Sampling Distribution of $\sqrt{n}(\bar{X}_n - \mu)/S$?

From slide 8:

$$\frac{\bar{X}_n - \mu}{S/\sqrt{n}} = \frac{\left(\dfrac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)}{\sqrt{\left[\frac{(n-1)S^2}{\sigma^2}\right]/(n-1)}}$$

$$= \frac{N(0,1)}{\sqrt{\dfrac{\chi^2(n-1)}{n-1}}}$$

$$\sim t(n-1)$$

This is the most important result of this lecture.

Strictly speaking, need to show that numerator and denominator are independent, but you can take my word for it!

Punchline: Sampling Distribution of $\sqrt{n}(\bar{X}_n - \mu)/S$

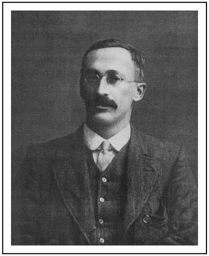
If $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$, then

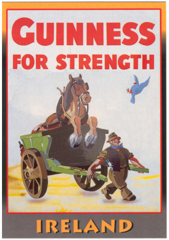
$$\boxed{\frac{\bar{X}_n - \mu}{S/\sqrt{n}} \sim t(n-1)}$$

We call this the 'student' t-distribution

Who was “Student?”

“Guinnessometrics: The Economic Foundation of Student’s t”





“Student” is the pseudonym used in 19 of 21 published articles by William Sealy Gosset, who was a chemist, brewer, inventor, and self-trained statistician, agronomer, and designer of experiments ... [Gosset] worked his entire adult life ... as an experimental brewer for one employer: Arthur Guinness, Son & Company, Ltd., Dublin, St. Jamess Gate. Gosset was a master brewer and rose in fact to the top of the top of the brewing industry: Head Brewer of Guinness.

Three Key Sampling Distributions

Suppose that $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$. Then:

$$\left(\frac{n-1}{\sigma^2}\right) S^2 \sim \chi^2(n-1)$$

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$\frac{\bar{X}_n - \mu}{S/\sqrt{n}} \sim t(n-1)$$

You should remember these three results, in particular the final two

CI for Mean of Normal Distribution, Popn. Var. Unknown

Same argument as we used when the variance was known, except with $t(n-1)$ rather than standard normal distribution:

$$P\left(-c \leq \frac{\bar{X}_n - \mu}{S/\sqrt{n}} \leq c\right) = 1 - \alpha$$

$$P\left(\bar{X}_n - c\frac{S}{\sqrt{n}} \leq \mu \leq \bar{X}_n + c\frac{S}{\sqrt{n}}\right) = 1 - \alpha$$

$c = \texttt{qt}(1 - \alpha/2, \texttt{df} = n - 1)$

$$\boxed{\bar{X}_n \pm \texttt{qt}(1 - \alpha/2, \texttt{df} = n - 1)\ \frac{S}{\sqrt{n}}}$$

Comparison of CIs for Mean of Normal Distribution

$100 \times (1-\alpha)\%$ Confidence Level

$$\boxed{X_1, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2)}$$

Known Population Std. Dev. (σ)

$$\bar{X}_n \pm \texttt{qnorm}(1-\alpha/2)\,\frac{\sigma}{\sqrt{n}}$$

Unknown Population Std. Dev. (σ)

$$\bar{X}_n \pm \texttt{qt}(1-\alpha/2, \texttt{df} = n-1)\,\frac{S}{\sqrt{n}}$$

Standard Error vs. Estimator of Standard Error

Standard Error

Recall that the standard deviation of the sampling distribution of an estimator is called the *standard error (SE)* of that estimator.

Example: Standard Error of the Mean

$SE(\bar{X}_n) = \sqrt{Var\left(\bar{X}_n\right)} = \sigma/\sqrt{n}$

Estimator of Standard Error of the Mean

Whereas $\sigma/\sqrt{n}$ *is* the standard error of the mean, $S/\sqrt{n}$ is an *estimator* of the standard error of the mean: $\widehat{SE}(\bar{X}_n) = S/\sqrt{n}$

Writing the CIs in terms of Actual and Estimated SE

$100 \times (1 - \alpha)\%$ Confidence Level

$$\boxed{X_1, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2)}$$

Known Population Std. Dev. (σ)

$$\bar{X}_n \pm \texttt{qnorm}(1 - \alpha/2)\, SE(\bar{X}_n)$$

Unknown Population Std. Dev. (σ)

$$\bar{X}_n \pm \texttt{qt}(1 - \alpha/2, \texttt{df} = n - 1)\, \widehat{SE}(\bar{X}_n)$$

Comparison of Normal and t CIs

Table: Values of $\texttt{qt}(1-\alpha/2, \texttt{df} = n-1)$ for various choices of n and α.

n	1	5	10	30	100	∞
$\alpha = 0.10$	6.31	2.02	1.81	1.70	1.66	1.64
$\alpha = 0.05$	12.71	2.57	2.23	2.04	1.98	1.96
$\alpha = 0.01$	63.66	4.03	3.17	2.75	2.63	2.58

Recall that as $n \to \infty$, $t(n-1) \to N(0,1)$

In a sense, using the t-distribution involves making a "small-sample correction." In other words, it is only when n is fairly small that this makes a practical difference for our confidence intervals.

Am I Taller Than The Average American Male?

Source: Centers for Disease Control (pg. 16)

Sample Mean	69 inches
Sample Std. Dev.	6 inches
Sample Size	5647
My Height	73 inches

$$\widehat{SE}(\bar{X}_n) = s/\sqrt{n} = 6/\sqrt{5647} \approx 0.08$$

Assuming the population is normal,

$$\bar{X}_n \pm \texttt{qt}(1-\alpha/2, \texttt{df} = n-1)\ \widehat{SE}(\bar{X}_n)$$

What is the approximate value of

`qt(1-0.05/2, df = 5646)`?

For large n, $t(n-1) \approx N(0,1)$, so the answer is approximately 2

What is the ME for the 95% CI?

$ME \approx 0.08 \times 2 = 0.16$

$\Rightarrow$ 95% CI $= 69 \pm 0.16$