Homework questions, week 3

Econ 103

1 Daily Homework questions

The questions in bold font are due on **Thursday 9th June**. You do not need to hand in the questions that are not in bold, though these will be useful to complete for your own understanding.

Lecture 8 - Discrete Random Variables II

Textbook questions:
Chapter 4: 7, **15**, 27, **29**

Solution: Solutions to textbook questions in back of textbook

Additional questions:

1. **Fill in the missing details from class to calculate the variance of a Bernoulli Random Variable *directly*, that is *without* using the shortcut formula. (Lecture 8, slide 11)**

Solution:

$$\begin{aligned}
\sigma^2 &= Var(X) = \sum_{x\in\{0,1\}} (x-\mu)^2 p(x) \\
&= \sum_{x\in\{0,1\}} (x-p)^2 p(x) \\
&= (0-p)^2(1-p) + (1-p)^2 p \\
&= p^2(1-p) + (1-p)^2 p \\
&= p^2 - p^3 + p - 2p^2 + p^3 \\
&= p - p^2 \\
&= p(1-p)
\end{aligned}$$

2. Prove that the Bernoulli Random Variable is a special case of the Binomial Random variable for which $n = 1$. (Hint: compare pmfs.)

Solution: The pmf for a Binomial(n, p) random variable is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

with support $\{0, 1, 2 \ldots, n\}$. Setting $n = 1$ gives,

$$p(x) = p(x) = \binom{1}{x} p^x (1-p)^{1-x}$$

with support $\{0, 1\}$. Plugging in each realization in the support, and recalling that $0! = 1$, we have

$$p(0) = \frac{1!}{0!(1-0)!} p^0 (1-p)^{1-0} = 1 - p$$

and

$$p(1) = \frac{1!}{1!(1-1)!} p^1 (1-p)^0 = p$$

which is exactly how we defined the Bernoulli Random Variable.

Lecture 9 - Discrete Random Variables III

Textbook questions:
Chapter 5: 1, 3, **5**, **11**, 13

Solution: Solutions to textbook questions in back of textbook

Additional questions:

1. **Suppose that X is a random variable with support $\{1, 2\}$ and Y is a random variable with support $\{0, 1\}$ where X and Y have the following joint distribution:**

$$p_{XY}(1,0) = 0.20, \quad p_{XY}(1,1) = 0.30$$
$$p_{XY}(2,0) = 0.25, \quad p_{XY}(2,1) = 0.25$$

   (a) Express the joint distribution in a 2×2 table.

Solution:

		X	
		1	2
Y	0	0.20	0.25
	1	0.30	0.25

(b) Using the table, calculate the marginal probability distributions of X and Y.

Solution:

$$\begin{aligned} p_X(1) &= p_{XY}(1,0) + p_{XY}(1,1) = 0.20 + 0.30 = 0.50 \\ p_X(2) &= p_{XY}(2,0) + p_{XY}(2,1) = 0.25 + 0.25 = 0.50 \\ p_Y(0) &= p_{XY}(1,0) + p_{XY}(2,0) = 0.20 + 0.25 = 0.45 \\ p_Y(1) &= p_{XY}(1,1) + p_{XY}(2,1) = 0.30 + 0.25 = 0.55 \end{aligned}$$

(c) Calculate the conditional probability distribution of $Y|X = 1$ and $Y|X = 2$.

Solution: The distribution of $Y|X = 1$ is

$$\begin{aligned} P(Y = 0|X = 1) &= \frac{p_{XY}(1,0)}{p_X(1)} = \frac{0.2}{0.5} = 0.4 \\ P(Y = 1|X = 1) &= \frac{p_{XY}(1,1)}{p_X(1)} = \frac{0.3}{0.5} = 0.6 \end{aligned}$$

while the distribution of $Y|X = 2$ is

$$\begin{aligned} P(Y = 0|X = 2) &= \frac{p_{XY}(2,0)}{p_X(2)} = \frac{0.25}{0.5} = 0.5 \\ P(Y = 1|X = 2) &= \frac{p_{XY}(2,1)}{p_X(2)} = \frac{0.25}{0.5} = 0.5 \end{aligned}$$

(d) Calculate $E[Y|X]$.

Solution:

$$\begin{aligned} E[Y|X = 1] &= 0 \times 0.4 + 1 \times 0.6 = 0.6 \\ E[Y|X = 2] &= 0 \times 0.5 + 1 \times 0.5 = 0.5 \end{aligned}$$

Hence,

$$E[Y|X] = \begin{cases} 0.6 & \text{with probability } 0.5 \\ 0.5 & \text{with probability } 0.5 \end{cases}$$

since $p_X(1) = 0.5$ and $p_X(2) = 0.5$.

(e) What is $E[E[Y|X]]$?

Solution: $E[E[Y|X]] = 0.5 \times 0.6 + 0.5 \times 0.5 = 0.3 + 0.25 = 0.55$. Note that this equals the expectation of Y calculated from its marginal distribution, since $E[Y] = 0 \times 0.45 + 1 \times 0.55$. This illustrates the so-called "Law of Iterated Expectations."

(f) Calculate the covariance between X and Y using the shortcut formula.

Solution: First, from the marginal distributions, $E[X] = 1 \cdot 0.5 + 2 \cdot 0.5 = 1.5$ and $E[Y] = 0 \cdot 0.45 + 1 \cdot 0.55 = 0.55$. Hence $E[X]E[Y] = 1.5 \cdot 0.55 = 0.825$. Second,

$$\begin{aligned} E[XY] &= (0 \cdot 1) \cdot 0.2 + (0 \cdot 2) \cdot 0.25 + (1 \cdot 1) \cdot 0.3 + (1 \cdot 2)0.25 \\ &= 0.3 + 0.5 = 0.8 \end{aligned}$$

Finally $Cov(X, Y) = E[XY] - E[X]E[Y] = 0.8 - 0.825 = -0.025$

2. Fill in the missing steps from lecture to prove the shortcut formula for covariance:

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

Solution: By the Linearity of Expectation,

$$\begin{aligned} Cov(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] \\ &= E[XY] - \mu_x E[Y] - \mu_Y E[X] + \mu_X \mu_Y \\ &= E[XY] - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y \\ &= E[XY] - \mu_X \mu_Y \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

3. (HARD) Let X_1 be a random variable denoting the returns of stock 1, and X_2 be a random variable denoting the returns of stock 2. Accordingly let $\mu_1 = E[X_1]$, $\mu_2 = E[X_2]$, $\sigma_1^2 = Var(X_1)$, $\sigma_2^2 = Var(X_2)$ and $\rho = Corr(X_1, X_2)$. A *portfolio*, Π, is a linear combination of X_1 and X_2 with weights that sum to one, that is $\Pi(\omega) = \omega X_1 + (1-\omega)X_2$, indicating the proportions of stock 1 and stock 2 that an investor holds. In this example, we require $\omega \in [0, 1]$, so that *negative* weights are not allowed. (This rules out short-selling.)

   (a) Calculate $E[\Pi(\omega)]$ in terms of ω, μ_1 and μ_2.

   Solution:

$$\begin{aligned} E[\Pi(\omega)] &= E[\omega X_1 + (1-\omega)X_2] = \omega E[X_1] + (1-\omega)E[X_2] \\ &= \omega\mu_1 + (1-\omega)\mu_2 \end{aligned}$$

   (b) If $\omega \in [0, 1]$ is it possible to have $E[\Pi(\omega)] > \mu_1$ *and* $E[\Pi(\omega)] > \mu_2$? What about $E[\Pi(\omega)] < \mu_1$ and $E[\Pi(\omega)] < \mu_2$? Explain.

   Solution: No. If short-selling is disallowed, the portfolio expected return must be between μ_1 and μ_2.

   (c) Express $Cov(X_1, X_2)$ in terms of ρ and σ_1, σ_2.

   Solution: $Cov(X, Y) = \rho\sigma_1\sigma_2$

   (d) What is $Var[\Pi(\omega)]$? (Your answer should be in terms of ρ, σ_1^2 and σ_2^2.)

   Solution:

$$\begin{aligned} Var[\Pi(\omega)] &= Var[\omega X_1 + (1-\omega)X_2] \\ &= \omega^2 Var(X_1) + (1-\omega)^2 Var(X_2) + 2\omega(1-\omega)Cov(X_1, X_2) \\ &= \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\rho\sigma_1\sigma_2 \end{aligned}$$

   (e) Using part (d) show that the value of ω that minimizes $Var[\Pi(\omega)]$ is

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

   In other words, $\Pi(\omega^*)$ is the *minimum variance portfolio.*

Solution: The First Order Condition is:

$$2\omega\sigma_1^2 - 2(1-\omega)\sigma_2^2 + (2-4\omega)\rho\sigma_1\sigma_2 = 0$$

Dividing both sides by two and rearranging:

$$\begin{aligned}\omega\sigma_1^2 - (1-\omega)\sigma_2^2 + (1-2\omega)\rho\sigma_1\sigma_2 &= 0\\ \omega\sigma_1^2 - \sigma_2^2 + \omega\sigma_2^2 + \rho\sigma_1\sigma_2 - 2\omega\rho\sigma_1\sigma_2 &= 0\\ \omega(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) &= \sigma_2^2 - \rho\sigma_1\sigma_2\end{aligned}$$

So we have

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

(f) If you want a challenge, check the second order condition from part (e).

Solution: The second derivative is

$$2\sigma_1^2 - 2\sigma_2^2 - 4\rho\sigma_1\sigma_2$$

and, since $\rho = 1$ is the largest possible value for ρ,

$$2\sigma_1^2 - 2\sigma_2^2 - 4\rho\sigma_1\sigma_2 \geq 2\sigma_1^2 - 2\sigma_2^2 - 4\sigma_1\sigma_2 = 2(\sigma_1 - \sigma_2)^2 \geq 0$$

so the second derivative is positive, indicating a minimum. This is a global minimum since the problem is quadratic in ω.

Lecture 10 - Continuous Random Variables I

Textbook questions: none

Additional questions:

1. **Suppose that X is a random variable with the following PDF**

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Graph the PDF of X.

Solution: It's an isosceles triangle with base from (0,0) to (2,0) and height 1.

(b) Show that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Solution:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\, dx &= \int_0^1 x\, dx + \int_1^2 (2-x)\, dx = \frac{x^2}{2}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2 \\ &= 1/2 + (4-2) - (2-1/2) = 1\end{aligned}$$

(c) What is $P(0.5 < X < 1.5)$?

Solution:

$$\begin{aligned}P(0.5 < X < 1.5) &= \int_{0.5}^{1.5} f(x)\, dx = \int_{0.5}^{1} x\, dx + \int_1^{1.5} (2-x)\, dx \\ &= \frac{x^2}{2}\Big|_{0.5}^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^{1.5} \\ &= (1/2 - 1/8) + (3 - 9/8) - (2 - 1/2) \\ &= 3/8 + 15/8 - 2 + 1/2 = 18/8 - 16/8 + 4/8 \\ &= 6/8 = 3/4 = 0.75\end{aligned}$$

2. **A random variable is said to follow a Uniform(a, b) distribution if it is equally likely to take on any value in the range $[a, b]$ and never takes a value outside this range. Suppose that X is such a random variable, i.e. $X \sim$ Uniform(a, b).**

(a) What is the support of X?

Solution: $[a, b]$

(b) Explain why the PDF of X is $f(x) = 1/(b - a)$ for $a \leq x \leq b$, zero elsewhere.

Solution: This simply generalizes the Uniform$(0, 1)$ random variable from class. To capture the idea that X is equally likely to take on any value in the range $[a, b]$, the PDF must be constant. To ensure that it integrates to 1, the denominator must be $b - a$.

(c) Using the PDF from part (b), calculate the CDF of X.

Solution:

$$F(x_0) \quad = \quad \int_{-\infty}^{x_0} f(x)\,dx = \int_a^{x_0} \frac{dx}{b-a} = \left.\frac{x}{b-a}\right|_a^{x_0} = \frac{x_0 - a}{b-a}$$

(d) Verify that $f(x) = F'(x)$ for the present example.

Solution:

$$F'(x) = \frac{d}{dx}\left(\frac{x-a}{b-a}\right) = \frac{1}{b-a} = f(x)$$

(e) Calculate $E[X]$.

Solution:

$$E[X] \quad = \quad \int_{-\infty}^{\infty} x f(x)\,dx = \int_a^b \frac{x}{b-a}\,dx = \left.\frac{x^2}{2(b-a)}\right|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

(f) Calculate $E[X^2]$. *Hint:* recall that $b^3 - a^3$ can be factorized as $(b-a)(b^2+a^2+ab)$.

Solution:

$$\begin{aligned} E[X^2] \quad &= \quad \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_a^b \frac{x^2}{b-a} = \left.\frac{x^3}{3(b-a)}\right|_a^b = \frac{b^3-a^3}{3(b-a)} \\ &= \quad \frac{(b-a)(b^2+a^2+ab)}{3(b-a)} = \frac{b^2+a^2+ab}{3} \end{aligned}$$

(g) Using the shortcut formula and parts (e) and (f), show that $Var(X) = (b-a)^2/12$.

Solution:

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 = \frac{b^2 + a^2 + ab}{3} - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{b^2 + a^2 + ab}{3} - \frac{a^2 + 2ab + b^2}{4} \\ &= \frac{4b^2 + 4a^2 + 4ab - 3a^2 - 6ab - 3b^2}{12} \\ &= \frac{b^2 + a^2 - 2ab}{12} = \frac{(b-a)^2}{12} \end{aligned}$$

Lecture 11 - Continuous Random Variables II

Textbook questions: Chapter 4: 19, **21, 23** (*When necessary, use R rather than the Normal tables in the front of the textbook.*)

Solution: Solutions to textbook questions in back of textbook

1. **Suppose that $X \sim N(0, 16)$ independent of $Y \sim N(2, 4)$. Recall that our convention is to express the normal distribution in terms of its mean and variance, i.e. $N(\mu, \sigma^2)$. Hence, X has a mean of zero and variance of 16, while Y has a mean of 2 and a variance of 4. In completing some parts of this question you will need to use the R function `pnorm` described in class. In this case, please write down the command you used as well as the numeric result.**

   (a) Calculate $P(-8 \leq X \leq 8)$.

   Solution:

   $$P(-8 \leq X \leq 8) = P(-8/4 \leq X/4 \leq 8/4) = P(-2 \leq Z \leq 2) \approx 0.95$$

   where Z is a standard normal random variable.

   (b) Calculate $P(0 \leq Y \leq 4)$.

   Solution:

   $$P(0 \leq Y \leq 4) = P\left(\frac{0-2}{2} \leq \frac{Y-2}{2} \leq \frac{4-2}{2}\right) = P(-1 \leq Z \leq 1) \approx 0.68$$

where Z is a standard normal random variable.

(c) Calculate $P(-1 \leq Y \leq 6)$.

Solution:

$$\begin{aligned}
P(-1 \leq Y \leq 6) &= P\left(\frac{-1-2}{2} \leq \frac{Y-2}{2} \leq \frac{6-2}{2}\right) \\
&= P(-1.5 \leq Z \leq 2) \\
&= \Phi(2) - \Phi(-1.5) \\
&= \texttt{pnorm(2) - pnorm(-1.5)} \\
&\approx 0.91
\end{aligned}$$

where Z is a standard normal random variable.

(d) Calculate $P(X \geq 10)$.

Solution:

$$\begin{aligned}
P(X \geq 10) &= 1 - P(X \leq 10) = 1 - P(X/4 \leq 10/4) = 1 - P(Z \leq 2.5) \\
&= 1 - \Phi(2.5) = 1 - \texttt{pnorm(2.5)} \\
&\approx 0.006
\end{aligned}$$

2 R Tutorials

You should complete R Tutorial #3 by **Thursday 9th June**.

R tutorials will be posted on Piazza, with solution code.