Homework questions, week 2

Econ 103

1 Daily Homework questions

The questions in bold font are due on **Thursday 2nd June**. You do not need to hand in the questions that are not in bold, though these will be useful to complete for your own understanding.

Lecture 4 - Basic Probability I

Chapter 3: **1**, 3, **5**

Solution: Solutions to textbook questions in back of textbook

Additional questions:

1. **Suppose you flip a fair coin twice.**
   (a) List all the basic outcomes in the sample space.

   Solution: $S = \{HH, HT, TT, TH\}$

   (b) Let A be the event that you get at least one head. List all the basic outcomes in A.

   Solution: $A = \{HH, HT, TH\}$

   (c) What is the probability of A?

   Solution: $P(A) = 3/4 = 0.75$

   (d) List all the basic outcomes in A^c.

> **Solution:** $A^c = \{TT\}$

(e) What is the probability of A^c?

> **Solution:** $P(A^c) = 1/4$

2. Suppose everyone in a class of one hundred students flips a fair coin five times.

   (a) What is the probability that John Smith, a particular student in the class, gets five heads in a row?

   > **Solution:** $(1/2)^5 = 1/32 \approx 0.03$

   (b) What is the probability that at least one person gets five heads in a row?

   > **Solution:** Use the complement rule: let A be the event that at least one person gets five heads in a row. Calculate the probability that no one gets 5 heads in a row as follows:
   >
   > $$P(A^c) = (1 - 1/2^5)^{100} = (31/32)^{100} \approx 0.04$$
   >
   > Hence the desired probability is about 0.96.

Lecture 5 - Basic Probability II

Chapter 3: **9, 13**

> **Solution:** Solutions to textbook questions in back of textbook

Additional questions:

1. **Suppose I deal two cards at random from a well-shuffled deck of 52 playing cards. What is the probability that I get a pair of aces?**

> **Solution:** You can either solve this assuming that order doesn't matter:
>
> $$\frac{\binom{4}{2}}{\binom{52}{2}} = \frac{4!/(2! \times 2!)}{52!/(50! \times 2!)} = \frac{6}{(52 \times 51)/2} = 6/1326 = 1/221$$

or that it does:

$$\frac{P_2^4}{P_2^{52}} = \frac{4!/2!}{52!/50!} = \frac{(4 \times 3)}{(52 \times 51)} = 12/2652 = 1/221$$

In either case, the answer is the same: $1/221 \approx 0.005$

2. (Adapted from Mosteller, 1965) A jury has three members: the first flips a coin for each decision, and each of the remaining two independently has probability p of reaching the correct decision. Call these two the "serious" jurors and the other the "flippant" juror (pun intended).

   (a) What is the probability that the serious jurors both reach the same decision?

   Solution: There are two ways for them to agree: they can either make the right decision, p^2, or the wrong decision, $(1-p)^2$. These are mutually exclusive, so we sum the probabilities for a total of $p^2 + (1-p)^2$

   (b) What is the probability that the serious jurors each reach different decisions?

   Solution: There are two ways for them to disagree: either the first makes the wrong decision, $p(1-p)$, or the second makes the wrong decision, $(1-p)p$. These are mutually exclusive, so we sum the probabilities for a total of $2p(1-p)$.

   (c) What is the probability that the jury reaches the correct decision? Majority rules.

   Solution: With probability p^2 the serious jurors agree and make the correct decision so the flippant juror is irrelevant. With probability $2p(1-p)$ they disagree. In half of these cases the flippant juror makes the correct decision. Thus, the overall probability is $p^2 + p(1-p) = p$.

Lecture 6 - Basic Probability III

Chapter 3: **11**, 15, **17**, 21, 23, 25, 27, **29 (hard)**

Solution: Solutions to textbook questions in back of textbook

Additional questions:

1. This question refers to the prediction market example from lecture. Imagine it is October 2012. Let O be a contract paying \$10 if Obama wins the election, zero otherwise, and R be a contract paying \$10 if Romney wins the election, zero otherwise. Let Price(O) and Price(R) be the respective prices of these contracts.

   (a) Suppose you *buy* one of each contract. What is your profit?

   Solution: Regardless of whether Romney or Obama wins, you get \$10. Thus, your profit is

   $$10 - \text{Price}(O) - \text{Price}(R)$$

   (b) Suppose you *sell* one of each contract. What is your profit?

   Solution: Regardless of whether Romney or Obama wins, you have to pay out \$10. Thus, your profit is

   $$\text{Price}(O) + \text{Price}(R) - 10$$

   (c) What must be true about Price(O) and Price(R), to prevent an opportunity for statistical arbitrage?

   Solution: From (a) we see that you can earn a guaranteed, risk-free profit from *buying* one of each contract whenever $10 > \text{Price}(O) + \text{Price}(R)$. From (b) we see that you can earn a guaranteeed, risk-free profit by *selling* one of each contract whenever $\text{Price}(O) + \text{Price}(R) > 10$. Therefore, the only way to prevent statistical arbitrage is to have $\text{Price}(O) + \text{Price}(R) = 10$.

   (d) How is your answer to part (c) related to the Complement Rule?

   Solution: In class we discussed how the market price of a prediction contract can be viewed as a subjective probability assessment. To find the implied probability we divide the price of the contract by the amount that is pays out, in this case \$10. Hence, dividing through by \$10, we see that the condition from part (b) when stated in probability terms is

   $$P(O) = 1 - P(R)$$

   This is precisely the Complement Rule because $R = O^c$.

   (e) What is the implicit assumption needed for your answers to parts (a)–(c) to be correct? How would your answers change if we were to relax this assumption?

Solution: The above discussion assumes that the only possible outcomes are Obama or Romney winning the election, that is $O \cup R = S$. This is equivalent to assuming that the probability of a third-party candidate winning the election is zero. If this assumption is not true, we need to redo the above with an extra contract. Let I be a contract that pays out \$10 if a third-party (i.e. independent) candidate wins the election, zero otherwise. Then the answers to the above become:

1. $10 - \text{Price}(O) - \text{Price}(R) - \text{Price}(I)$
2. $\text{Price}(O) + \text{Price}(R) + \text{Price}(I) - 10$
3. $\text{Price}(O) + \text{Price}(R) + \text{Price}(I) = 10$
4. The Complement Rule becomes:

$$P(I) = 1 - P(O) - P(R)$$

2. "Odd Question", from Hacking (2001) [NB - this is very hard]:

> You are a physician. You think it is quite likely that one of your patients has strep throat, but you aren't sure. You take some swabs from the throat and send them to a lab for testing. The test is (like nearly all lab tests) not perfect. If the patient has strep throat, then 70% if the time the lab says yes. But 30% of the time it says NO. If the patient does not have strep throat, then 90% of the time the lab says NO. But 10% of the time it says YES. You send five successive swabs to the lab, from the same patient. and get back these results in order: YES, NO, YES, NO, YES.

Let S be the event that the patient has strep throat, and S^c be the even that she does not. Let Y be the event that a given test says YES and $N = Y^c$ be the event that a given test says NO. You may assume that the tests are independent.

(a) Calculate the probability that your patient has strep throat. (Hint, there is a missing piece of information and you should express your answer *in terms of it*.)

Solution: The probabilities from the question statement are:

$$\begin{aligned} P(Y|S) &= 0.7 \\ P(N|S) &= 0.3 \\ P(Y|S^c) &= 0.1 \\ P(N|S^c) &= 0.9 \end{aligned}$$

We are asked to calculate $P(S|YNYNY)$ where $YNYNY$ denotes the sequence of outcomes YES, NO, YES, NO, YES *in that order* from the fives tests. By Bayes' Rule,

$$P(S|YNYNY) = \frac{P(YNYNY|S)P(S)}{P(YNYNY)}$$

From the information provided above, we can calculate everything *except* the base rate, so we will express everything in terms of $P(S)$. By independence,

$$\begin{aligned} P(YNYNY|S) &= P(Y|S) \times P(N|S) \times P(Y|S) \times P(N|S) \times P(Y|S) \\ &= P(Y|S)^3 \times P(N|S)^2 = (7/10)^3 \times (3/10)^2 \\ &= 343/1000 \times 9/100 \end{aligned}$$

and similarly,

$$\begin{aligned} P(YNYNY|S^c) &= P(Y|S^c) \times P(N|S^c) \times P(Y|S^c) \times P(N|S^c) \times P(Y|S^c) \\ &= P(Y|S^c)^3 \times P(N|S^c)^2 = (1/10)^3 \times (9/10)^2 \\ &= 1/1000 \times 81/100 \end{aligned}$$

Now, by the law of total probability,

$$\begin{aligned} P(YNYNY) &= P(YNYNY|S)P(S) + P(YNYNY|S^c)P(S^c) \\ &= P(YNYNY|S) \times P(S) + P(YNYNY|S^c) \times (1 - P(S)) \\ &= 343/1000 \times 9/100 \times P(S) \\ &\quad +1/1000 \times 81/100 \times (1 - P(S)) \end{aligned}$$

Therefore, multiplying the numerator and denominator by 10^5

$$\begin{aligned} P(S|YNYNY) &= \frac{P(YNYNY|S)P(S)}{P(YNYNY)} \\ &= \frac{343 \times 9 \times P(S)}{343 \times 9 \times P(S) + 81 \times (1 - P(S))} \\ &= \frac{3087P(S)}{3087P(S) + 81 - 81P(S)} \\ &= \frac{3087P(S)}{3006P(S) + 81} \\ &= \frac{3087}{3006 + 81/P(S)} \end{aligned}$$

(b) Based on your answer to part (b) do you think the patient has strep throat? Explain.

Solution: Since we don't know the base rate $P(S)$ we can't get an explicit value for the conditional probability from part (a). One way forward would be to ask what bounds on $P(S)$ would ensure that $P(S|YNYNY) > 1/2$. To do this, we solve the following expression for $P(S)$:

$$\begin{aligned} \frac{1}{2} &= \frac{3087}{3006 + 81/P(S)} \\ 3006 + 81/P(S) &= 6174 \\ 6174P(S) &= 3006P(S) + 81 \\ 3168P(S) &= 81 \\ P(S) &= 81/3168 \approx 0.026 \end{aligned}$$

Therefore, as long as $P(S) > 0.026$, the test results given above make it more likely than not that our patient has strep throat. But where does this leave us? How can we evaluate whether this restriction on the base rate is likely to be satisfied? In this example the term "base rate" is perhaps a little misleading as the relevant probability $P(S)$ is *not* the overall rate of strep throat in the population. A better term in this case would be "prior probability." That is, how likely is it *before we see that test results* that this patient has strep throat? The question statement says that you, the physician, think it is "quite likely" that the patient has strep throat, presumably based on her symptoms, etc. Perhaps "quite likely" should be interpreted as $P(S) = 0.9$, in which case $P(S|YNYNY) \approx 0.997$. I would certainly interpret "quite likely" as $P(S) > 1/2$ and if $P(S) = 1/2$, we have $P(S|YNYNY) \approx 0.974$. The following is a reasonable summary of our results. If you have *any reason to believe* a priori that your patient has strep throat, these test results imply that it is *extremely likely* she does. However, if you were to randomly test someone off the street who had no symptoms of strep throat and get the above results, the evidence would be much less convincing. I would doubt, for example, that more than 2.6% of the population have strep throat at any given time, as would be required to make the conditional probability greater than 1/2.

Lecture 7 - Discrete Random Variables I

Chapter 4: 1, 5, **9, 11, 25**

1. **Suppose X is a random variable with support $\{-1, 0, 1\}$ where $p(-1) = q$ and**

$p(1) = p$.

(a) What is $p(0)$?

> **Solution:** By the complement rule $p(0) = 1 - p - q$.

(b) Calculate the CDF, $F(x_0)$, of X.

> **Solution:**
> $$F(x_0) = \begin{cases} 0, \ x_0 < -1 \\ q, \ -1 \leq x_0 < 0 \\ 1 - p, \ 0 \leq x_0 < 1 \\ 1, \ x_0 \geq 1 \end{cases}$$

(c) Calculate $E[X]$.

> **Solution:** $E[X] = -1 \cdot q + 0 \cdot (1 - p - q) + p \cdot 1 = p - q$

(d) What relationship must hold between p and q to ensure $E[X] = 0$?

> **Solution:** $p = q$

2 R Tutorials

You should complete R Tutorial #2 by **Thursday 2nd June**.

R tutorials will be posted on Piazza, with solution code.